Elect. Comm. in Probab. **14** (2009), 66–80

ELECTRONIC COMMUNICATIONS in PROBABILITY

BROWNIAN COUPLINGS, CONVEXITY, AND SHY-NESS

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Submitted 30 September 2008, *accepted in final form* 20 January 2009

AMS 2000 Subject classification: 60J65
Keywords: Brownian motion; co-adapted coupling; convexity; coupling; Markovian coupling; perverse coupling; reflected Brownian motion; reflection coupling; shy coupling; Skorokhod construction; stochastic differential

Abstract

Benjamini, Burdzy, and Chen (2007) introduced the notion of a *shy coupling*: a coupling of a Markov process such that, for suitable starting points, there is a positive chance of the two component processes of the coupling staying at least a given positive distance away from each other for all time. Among other results, they showed that no shy couplings could exist for reflected Brownian motions in C^2 bounded convex planar domains whose boundaries contain no line segments. Here we use potential-theoretic methods to extend this Benjamini et al. (2007) result (a) to all bounded convex domains (whether planar and smooth or not) whose boundaries contain no line segments, (b) to all bounded convex planar domains regardless of further conditions on the boundary.

1 Introduction

Motivated by the use of reflection couplings for reflected Brownian motion in a number of contexts (for example efficient coupling as in Burdzy and Kendall, 2000, and work related to the hotspot conjecture, as in Atar and Burdzy, 2002, and recent work on Laugesen-Morpurgo conjectures, Pascu and Gageonea, 2008), Benjamini, Burdzy, and Chen (2007) introduced the notion of shy coupling and studied it in the contexts of Brownian motion on graphs and reflected Brownian motion particularly in convex domains. Shy coupling of two random processes occurs when, for suitable starting points, there is a positive chance of the two component processes of the coupling staying at least a given positive distance away from each other: this of course is in direct contrast to the more usual use of couplings, in which the objective is to arrange for the two processes to meet. Interest in shy couplings is therefore focused on characterizing situations in which shy coupling *cannot* occur. Benjamini et al. (2007) show non-existence of shy coupling for reflected Brownian motions in C^2 bounded convex planar domains whose boundaries contain no line segments. The present note (influenced by techniques used to study non-confluence of Γ-martingales in Kendall, 1990) uses rather direct potential-theoretic methods to obtain significantly stronger results in the case of reflected Brownian motion in convex domains (Theorems 7 and 8 below). As with Benjamini et al. (2007), the principal purpose is to contribute to a better understanding of probabilistic coupling.

To fix notation, we begin with a formal discussion of coupling and shy coupling. In the following D is a measurable space, and the distribution of a Markov process on D is specified as a semigroup of transition probability measures $\{P_t^z : t \geq 0, z \in D\}$. (The semigroup-based definition allows us to take account of varying starting points, a typical feature of coupling arguments.)

Definition 1 (Co-adapted coupling)**.** A *coupling* of a Markov process $\{P_t^z : t \geq 0, z \in D\}$ on a measurable space D is a family of random processes (X, Y) on D^2, one process (X, Y) for each pair of starting points $(x_0, y_0) \in D^2$, such that X and Y each separately are Markov but share the same semigroup of transition probability measures $\{P_t^z : t \geq 0, z \in D\}$. Thus for each $s, t \geq 0$ and $z \in D$ and each measurable $A \subseteq D$ we have

$$\begin{aligned} \mathbb{P}\left[X_{s+t} \in A \mid X_s = z,\ X_u : 0 \leq u \leq s\right] &= P_t^z(A)\,, \\ \mathbb{P}\left[Y_{s+t} \in A \mid Y_s = z,\ Y_u : 0 \leq u \leq s\right] &= P_t^z(A)\,. \end{aligned} \tag{1}$$

The coupling is said to be a *co-adapted coupling* if the conditioning in (1) can in each case be extended to include the pasts of both X and Y: for each $s, t \geq 0$ and $z \in D$ and each measurable $A \subseteq D$

$$\begin{aligned} \mathbb{P}\left[X_{s+t} \in A \mid X_s = z,\ X_u, Y_u : 0 \leq u \leq s\right] &= P_t^z(A)\,, \\ \mathbb{P}\left[Y_{s+t} \in A \mid Y_s = z,\ X_u, Y_u : 0 \leq u \leq s\right] &= P_t^z(A)\,. \end{aligned} \tag{2}$$

In contrast to Benjamini et al. (2007)'s notion of Markovian coupling, we do not require (X, Y) to be Markov. (This generalization is convenient but unimportant.) Note that the couplings in Benjamini et al. (2007) are all Markovian and hence co-adapted.

We say that (X, Y) begun at (x_0, y_0) *couples successfully* on the event

$$[X_t = Y_t \text{ for all sufficiently large } t]\,.$$

Note that a co-adapted coupling can be adjusted on the simpler event $[X_t = Y_t \text{ for some } t]$ so as to couple successfully on the new event; this need not be the case for more general couplings. In the remainder of this note the term "coupling" will always be short for "co-adapted coupling".

Benjamini et al. (2007)'s notion of *shy coupling* for a Markov process on D is primarily concerned with the cases of Brownian motion on graphs and reflected Brownian motion on domains in Euclidean space. However their definition is expressed in general terms: suppose that D is actually a metric space equipped with a distance dist which furnishes a Borel measurability structure:

Definition 2 (Shy coupling)**.** A coupling (X, Y) is *shy* if there exist two distinct starting points $x_0 \neq y_0$ such that for some $\varepsilon > 0$

$$\mathbb{P}\left[\operatorname{dist}(X_t, Y_t) > \varepsilon \text{ for all } t \mid X_0 = x_0, Y_0 = y_0\right] \quad > \quad 0\,. \tag{3}$$

In words, X, Y has positive chance of failing to ε-couple for some $\varepsilon > 0$ and some pairs of starting points.

We say that (X, Y) begun at (x_0, y_0) *ε-couples* (for $\varepsilon > 0$) on the event

$$[\operatorname{dist}(X_t, Y_t) = \varepsilon \text{ for some } t]\,.$$

We focus on reflecting Brownian motion in a bounded convex domain in finite-dimensional Euclidean space:

Definition 3 (Reflecting Brownian motion)**.** A reflecting Brownian motion in the closure $\overline{D}$ of a bounded convex domain $D \subset \mathbb{R}^n$, begun at $x_0 \in D$, is a Markov process X in $\overline{D}$ solving

$$X_t \quad = \quad x_0 + B_t + \int_0^t \nu(X_s)\,\mathrm{d}\,L_s^X\,, \tag{4}$$

where B is n-dimensional standard Brownian motion, L^X measures local time of X accumulated at the boundary ∂D, and ν is a choice of an inward-pointing unit normal vectorfield on ∂D.

Unique solutions of (4) exist in the case of bounded convex domains (Tanaka, 1979, Theorem 3.1); this is a consequence of results about the deterministic Skorokhod's equation. Note that reflected Brownian motion can be defined as a semimartingale for domains which are much more general than convex domains (see for example the treatment of Lipschitz domains in Bass and Hsu, 1990); however the results of this paper are concerned entirely with the convex case.
Recall that a convex domain in Euclidean spaces can be viewed as the intersection of countably many half-spaces. Consequently the inward-pointing unit normal vectorfield ν is unique up to a subset of $\partial D = \overline{D} \setminus D$ of zero Hausdorff $(n-1)$-measure. The local time term L^X can be defined using the Skorokhod construction: it is the minimal non-decreasing process such that the solution of (4) stays confined within $\overline{D}$. It is determined by the choice of normal vectorfield ν.
Arguments from stochastic calculus show that all co-adapted couplings of such reflected Brownian motions can be represented in terms of co-adapted couplings (A, B) of n-dimensional standard Brownian motions, which in turn must satisfy

$$A_t \quad = \quad \int_0^t \mathbb{J}_s^\top\,\mathrm{d}\,B_s + \int_0^t \mathbb{K}_s^\top\,\mathrm{d}\,C_s\,,$$

where C is an n-dimensional standard Brownian motion independent of B, and $\mathbb{J}$, $\mathbb{K}$ are $(n\times n)$-matrix-valued random processes, adapted to the filtration generated by B and C, and satisfying the identity

$$\mathbb{J}^\top\,\mathbb{J} + \mathbb{K}^\top\,\mathbb{K} \quad = \quad \mathbb{I}^{(n)} \text{ (the identity matrix on } \mathbb{R}^n)\,.$$

A proof of this fact can be found in passing on page 297 of Émery (2005). An explicit statement and a slightly more direct proof is to be found below at Lemma 6.
Thus we will study X and Y such that

$$\begin{aligned}
X_t \quad &= \quad x_0 + B_t + \int_0^t \nu(X_s)\,\mathrm{d}\,L_s^X\,,\\
Y_t \quad &= \quad y_0 + \int_0^t \mathbb{J}_s^\top\,\mathrm{d}\,B_s + \int_0^t \mathbb{K}_s^\top\,\mathrm{d}\,C_s + \int_0^t \nu(Y_s)\,\mathrm{d}\,L_s^Y\,,\\
\mathbb{J}_t^\top\,\mathbb{J}_t + \mathbb{K}_t^\top\,\mathbb{K}_t \quad &= \quad \mathbb{I}^{(n)}\,.
\end{aligned} \tag{5}$$

The results of this note concern all possible co-adapted couplings, of which two important examples are:

1. the *reflection coupling*:
   $\mathbb{K} = 0$ and $\mathbb{J}_t = \mathbb{I}^{(n)} - 2\,\mathbf{e}_t\,\mathbf{e}_t^\top$, where $\mathbf{e}_t = (X_t - Y_t)/\|X_t - Y_t\|$, so that $\mathrm{d}A$ is the reflection of $\mathrm{d}B$ in the hyperplane bisecting the segment from X_t to Y_t;

   and its opposite,

2. the *perverse coupling*:
$\mathbb{K} = 0$ and $\mathbb{J}_t = -\mathbb{I}^{(n)} + 2\,\mathbf{e}_t\,\mathbf{e}_t^\top$, so that the distance $\|X - Y\|$ has trajectories purely of locally bounded variation. (This is related to the uncoupled construction in Émery, 1989, Exercise 5.43 of the planar process Z'' with deterministic radial part.)

In practice, as is typically the case when studying Brownian couplings, general proofs follow easily from the special case when $\mathbb{K} = 0$ and $\mathbb{J}$ is an orthogonal matrix. (Heuristic remarks related to this observation are to be found in Kendall, 2007, Section 2.)
Note that the existence question for reflection couplings in convex domains is non-trivial (Atar and Burdzy 2004 establish existence for the more general case of lip domains), but is not relevant for our purposes.
Benjamini et al. (2007) use ingenious arguments based on ideas from differential games to show that a bounded convex planar domain cannot support any shy couplings of reflected Brownian motions if the boundary is C^2 and contains no line segments (Benjamini et al., 2007, Theorem 4.3). Here we describe the use of rather direct potential-theoretic methods (summarized in section 2); in section 3 these are used to generalize the Benjamini et al. (2007) result to the cases of (a) all bounded convex domains in Euclidean space of whatever dimension with boundaries which need not be smooth but must contain no line segments (Theorem 7), and (b) all bounded convex planar domains (Theorem 8). Further extensions and conjectures are discussed in section 4.

2 Some lemmas from stochastic calculus and potential theory

The following two lemmas from probabilistic potential theory are fundamental for our approach.

Lemma 4. *For $D \subset \mathbb{R}^n$ a bounded domain, for fixed $\varepsilon > 0$, consider the closed and bounded (and therefore compact) subset of $\mathbb{R}^n \times \mathbb{R}^n$ given by*

$$F \quad = \quad (\overline{D} \times \overline{D}) \setminus \{(x, y) : \operatorname{dist}(x, y) < \varepsilon\}\,.$$

Suppose there is a continuous function $\Psi : F \to \mathbb{R}$ such that the following random process Z is a supermartingale for any *co-adapted coupling of reflecting Brownian motions X and Y in the closure $\overline{D}$, when S is the exit time of (X, Y) from F:*

$$Z_t \quad = \quad \Psi(X_{t\wedge S}, Y_{t\wedge S}) + t \wedge S\,.$$

Then for any such coupling almost surely $S < \infty$ and $\operatorname{dist}(X_S, Y_S) = \varepsilon$.

Proof. Being a continuous function on a compact set F, Ψ must be bounded. Thus for any co-adapted coupling (X, Y) the supermartingale Z is bounded below, and so almost surely it must converge to a finite value at time ∞. But boundedness of Ψ also means that Z must almost surely tend to infinity on the event $[S = \infty]$. These two requirements on Z force the conclusion that $\mathbb{P}[S < \infty] = 1$. Since X and Y each reflect off ∂D, it follows that the exit point (X_S, Y_S) must belong to the part of ∂F which is contained in the boundary of $\{(x, y) : \operatorname{dist}(x, y) < \varepsilon\}$; hence almost surely $\operatorname{dist}(X_S, Y_S) = \varepsilon$. $\square$

Consequently, if one can exhibit such a Ψ for a specified D then any coupled X, Y in $\overline{D}$ must ε-couple. Indeed the existence of such a Ψ implies a uniform bound on exponential moments of the ε-coupling time S.

As implied by Benjamini et al. (2007, Example 4.2), the obvious possibility $\Psi = c\|X-Y\|^\alpha$ (for positive c, α with α small) is not suitable here; the perverse coupling of the previous section is an example for which all such Ψ lead to $\Psi(X_{t\wedge S}, Y_{t\wedge S}) + t\wedge S$ being a strictly increasing process when neither X nor Y belong to ∂D. Nevertheless we will see in the next section how to construct Ψ which work simultaneously for all co-adapted couplings in a wide range of bounded convex domains.

Before embarking on this, it is convenient to introduce a further lemma motivated by Itô (1975)'s approach to stochastic calculus using modules of stochastic differentials: if $Z = M + A$ is the Doob-Meyer decomposition of a semimartingale Z into local martingale M and process A of locally bounded variation, then write $\operatorname{Drift} \mathrm{d}Z = \mathrm{d}A$ and write $(\mathrm{d}Z)^2 = \mathrm{d}[M,M]$ for the differential of the (increasing) bracket process of M.

Lemma 5. *The conclusion of Lemma 4 holds if there is a continuous $\Phi : F \to \mathbb{R}$ such that, for all co-adapted couplings (X,Y) as above,*

> $Z = \Phi(X,Y)$ *is a semimartingale;*
>
> *moreover the stochastic differential* $\mathrm{d}Z$ *satisfies the following* random measure *inequalities (with a, $b > 0$):*
>
> 1. ***(Volatility bounded from below)*** $(\mathrm{d}Z)^2 > a\,\mathrm{d}t$ *up to time S;*
> 2. ***(Drift bounded from above)*** $\operatorname{Drift}\mathrm{d}Z < b\,\mathrm{d}t$ *up to time S.*

Proof. For $c, \lambda > 0$ to be chosen at the end of the proof, set $\Psi(x,y) = c(1-\exp(-\lambda\Phi(x,y)))$ and apply Itô's lemma to $\Psi(X,Y) = c(1-\exp(-\lambda Z))$ up to the exit time S:

$$\begin{aligned}\operatorname{Drift}\mathrm{d}\Psi(X,Y) &= c\lambda\exp(-\lambda Z)\left(\operatorname{Drift}\mathrm{d}Z - \tfrac{1}{2}\lambda(\mathrm{d}Z)^2\right)\\ &\leq -\tfrac{c}{2}\lambda\exp(-\lambda Z)(\lambda a - 2b)\,\mathrm{d}t\end{aligned}$$

(where the inequality is viewed as an inequality for random measures). Fix $\lambda > 2b/a$ and $c > 2\lambda^{-1}\frac{\exp(\lambda\max\{\Phi\})}{\lambda a - 2b}$. The above calculation then shows that $\Psi(X_{t\wedge S}, Y_{t\wedge S}) + (t\wedge S)$ is a supermartingale, so the conclusion of Lemma 4 holds. □

As mentioned above in section 1, the following stochastic calculus result is to be found unannounced and in passing on page 297 of Émery (2005). It provides an explicit representation for co-adaptively coupled Brownian motions. We state it here as a lemma and indicate a direct proof in order to establish an explicit statement of the result in the literature.

Lemma 6. *Suppose that A and B are two co-adapted n-dimensional Brownian motions. Augmenting the filtration if necessary by adding an independent adapted n-dimensional Brownian motion D, it is possible to construct an adapted n-dimensional Brownian motion C, independent of B, such that*

$$A = \int \mathbb{J}^\top \mathrm{d}B + \int \mathbb{K}^\top \mathrm{d}C\,,$$

where $\mathbb{J}$, $\mathbb{K}$ are $(n\times n)$-matrix-valued predictable random processes, satisfying the identity

$$\mathbb{J}^\top\,\mathbb{J} + \mathbb{K}^\top\,\mathbb{K} = \mathbb{I}^{(n)} \text{ (the identity matrix on } \mathbb{R}^n)\,.$$

Proof. Certainly the quadratic covariation between the vector semimartingales A and B may be expressed as $\mathrm{d}A\,\mathrm{d}B^\top = \mathbb{J}^\top \mathrm{d}\,t$ for a predictable $(n\times n)$-matrix-valued predictable random process $\mathbb{J}$ such that the symmetric matrix inequality $0 \leq \mathbb{J}^\top \mathbb{J} \leq \mathbb{I}^{(n)}$ holds in the spectral sense ($0 \leq x^\top \mathbb{J}^\top \mathbb{J}\, x \leq x^\top x$ for all vectors x); this is a consequence of the Kunita-Watanabe inequality. Since $\mathbb{J}^\top \mathbb{J}$ is a symmetric contraction matrix we find $(\mathbb{J}^\top \mathbb{J})^k \to \mathbb{H}_1$ as $k\to\infty$, where $\mathbb{H}_1$ is the orthogonal projection onto the null-space of $\mathbb{J}^\top \mathbb{J} - \mathbb{I}^{(n)}$. This exhibits $\mathbb{H}_1$ as a measurable function of $\mathbb{J}$. We may now extract $\lambda_2^2 = \sup\{x^\top(\mathbb{J}^\top \mathbb{J} - \mathbb{H}_1)x : \|x\| = 1\}$ as a measurable function of $\mathbb{J}$, renormalize and study $(\mathbb{J}^\top \mathbb{J} - \mathbb{H}_1)^k/\lambda_2^{2k} \to \mathbb{H}_2$, and continue so as to represent

$$\mathbb{J}^\top \mathbb{J} \quad = \quad \mathbb{H}_1 + \sum_{i=2}^{N} \lambda_i^2\, \mathbb{H}_i\,, \tag{6}$$

where $\mathbb{H}_1$, $\mathbb{H}_2$, ..., $\mathbb{H}_N$ are disjoint orthogonal projections, $N \leq n$, and $1 > \lambda_2 > \ldots > \lambda_N > 0$ and all quantities are measurable functions of $\mathbb{J}$.
Set $\mathbb{H}_0 = \mathbb{I}^{(n)} - \mathbb{H}_1 - \mathbb{H}_2 - \ldots - \mathbb{H}_N$ and define

$$\mathbb{K} \quad = \quad \mathbb{H}_0 + \sum_{i=2}^{N} \sqrt{1-\lambda_i^2}\, \mathbb{H}_i$$

as the non-negative symmetric square root of $\mathbb{I}^{(n)} - \mathbb{J}^\top \mathbb{J}$. The above spectral approach defines $\mathbb{K}$ as a measurable function of the matrix $\mathbb{J}$, and hence we may view $\mathbb{K}$ as well as $\mathbb{J}$ as predictable $(n\times n)$-matrix-valued predictable random processes, now satisfying $\mathbb{J}^\top \mathbb{J} + \mathbb{K}^\top \mathbb{K} = \mathbb{I}^{(n)}$.
Moreover we may construct a pseudo-inverse of $\mathbb{K}$ as a further predictable symmetric process:

$$\mathbb{K}^+ \quad = \quad \mathbb{H}_0 + \sum_{i=2}^{N} \frac{1}{\sqrt{1-\lambda_i^2}}\, \mathbb{H}_i\,,$$

so that $\mathbb{K}^+\,\mathbb{K} = \mathbb{K}\,\mathbb{K}^+ = \mathbb{I}^{(n)} - \mathbb{H}_1$.
Now define C by

$$\mathrm{d}\,C \quad = \quad \mathbb{K}^+(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B) + \mathbb{H}_1\,\mathrm{d}D\,.$$

The quadratic variation of $\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B$ is

$$(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)^\top \quad = \quad (\mathbb{I}^{(n)} - \mathbb{J}^\top \mathbb{J})\,\mathrm{d}\,t = \mathbb{K}^\top \mathbb{K}\,\mathrm{d}\,t\,,$$

and so the stochastic differential $\mathbb{K}^+(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)$ has finite quadratic variation $(\mathbb{I}^{(n)} - \mathbb{H}_1)\,\mathrm{d}\,t$ and therefore (by the L^2 theory of stochastic differentials of continuous semimartingales) it is a martingale differential. Hence C is a continuous martingale with quadratic variation $\mathrm{d}\,C\;\mathrm{d}\,C^\top = \mathbb{I}^{(n)}\,\mathrm{d}\,t$, by which we may deduce that C is n-dimensional Brownian motion. Moreover $\mathrm{d}\,C\;\mathrm{d}B^\top = \mathbb{K}^+(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)\,\mathrm{d}B^\top = \mathbb{K}^+(\mathbb{J}^\top - \mathbb{J}^\top)\,\mathrm{d}\,t = 0$, so C is independent of B.
Finally $\mathbb{K}^\top \mathrm{d}\,C = (\mathbb{I}^{(n)} - \mathbb{H}_1)(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)$ (use $\mathbb{K}^\top \mathbb{H}_1 = 0$) and $\mathbb{H}_1(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)$ is a martingale differential with quadratic variation

$$\begin{aligned}\mathbb{H}_1(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)(\mathrm{d}A - \mathbb{J}^\top \mathrm{d}B)^\top \mathbb{H}_1 \quad &= \quad \mathbb{H}_1(\mathbb{I}^{(n)} - \mathbb{J}^\top \mathbb{J})\mathbb{H}_1\,\mathrm{d}\,t \\ &= \quad \mathbb{H}_1\,\mathbb{K}^\top \mathbb{K}\,\mathbb{H}_1\,\mathrm{d}\,t \quad = \quad 0\,.\end{aligned}$$

So $\mathbb{K}^\top \mathrm{d}\,C = \mathrm{d}A - \mathbb{J}^\top \mathrm{d}B$, which establishes the result. □

Émery (2005) uses an approximation argument, which we circumvent here by exhibiting $\mathbb{K}$ as a predictable process via the spectral decomposition (6). Note also that the extra Brownian motion D is not required if $\mathbb{J}^\top \mathbb{J} < \mathbb{I}^{(n)}$, in which case $\mathbb{H}_1 = 0$.

3 Theorems and proofs

The first theorem generalizes the planar result of Benjamini et al. (2007, Theorem 4.3) to the case of higher dimensions, and removes the requirement of boundary smoothness.

Theorem 7. *Let $\overline{D}$ be the closure of a bounded convex domain in $\mathbb{R}^n$ such that ∂D contains no line segments. Then no co-adapted coupling of reflected Brownian motions in $\overline{D}$ can be shy.*

Proof. It suffices to exhibit, for any fixed ε, a function Φ satisfying the two requirements of Lemma 5. Motivated by a similar construction used to establish the convex geometry of small hemispheres (Kendall, 1991), define (for $p \notin D$ and $\delta > 0$)

$$V_{p,\delta}(x,y) \;=\; \tfrac{1}{2}\|x-y\|^2 + \tfrac{\delta}{2}\|x-p\|^2 - \tfrac{\delta}{2}\|y-p\|^2 \;=\; \tfrac{1}{2}\|x-y\|^2 + \delta\langle x-y, \tfrac{x+y}{2} - p\rangle\,. \quad (7)$$

Thus $V_{p,\delta}$ is a hyperbolic perturbation of $\frac{1}{2}\|x-y\|^2$ based on the *pole* p. For any fixed $\varepsilon > 0$ and for all sufficiently small $\delta > 0$ depending on ε, p, and the geometry of D we show that $\Phi = V_{p,\delta}$ satisfies the requirements of Lemma 5. The result then follows by applying Lemmas 5 then 4.

The first step is to establish Lemma 5 requirement 1 **(volatility bounded from below)**. Fixing $\varepsilon > 0$, suppose by virtue of Lemma 6 that X, Y satisfy (5) for some co-adapted matrix processes $\mathbb{J}$, $\mathbb{K}$. Applying Itô's lemma and the fact that B and C are independent n-dimensional standard Brownian motions,

$$\begin{aligned}
(\mathrm{d}\,\Phi(X,Y))^2 \quad &= \\
&\Big(\|X-Y+\delta(X-p) - \mathbb{J}(X-Y+\delta(Y-p))\|^2 + \|\,\mathbb{K}(X-Y+\delta(Y-p))\|^2\Big)\,\mathrm{d}\,t \\
&\geq \quad \|X-Y+\delta(X-p) - \mathbb{J}(X-Y+\delta(Y-p))\|^2\,\mathrm{d}\,t \\
&\geq \quad \big(\|X-Y+\delta(X-p)\| - \|X-Y+\delta(Y-p)\|\big)^2\,\mathrm{d}\,t\,, \quad (8)
\end{aligned}$$

since the third equation of (5) implies that the linear map $\mathbb{J}$ is a contraction. Suppose now that $\|X-Y\| > \varepsilon$. If (for example)

$$\delta \quad < \quad \frac{\varepsilon}{2\sup\{\mathrm{dist}(p,w) : w \in D\}} \quad (9)$$

then

$$\begin{aligned}
\|X-Y+\delta(X-p)\|^2 \quad &= \quad \|X-Y+\delta(Y-p)+\delta(X-Y)\|^2 \\
&= \quad \|X-Y+\delta(Y-p)\|^2 + 2\delta\|X-Y\|^2 + 2\delta^2\langle X-Y, \tfrac{X+Y}{2} - p\rangle \\
&\geq \quad \|X-Y+\delta(Y-p)\|^2 + 2\delta\|X-Y\|^2 - 2\delta^2|\langle X-Y, \tfrac{X+Y}{2} - p\rangle| \\
&= \quad \|X-Y+\delta(Y-p)\|^2 + 2\delta\|X-Y\|^2\left(1 - \delta\frac{\left|\langle \frac{X-Y}{\|X-Y\|}, \frac{X+Y}{2} - p\rangle\right|}{\|X-Y\|}\right) \\
&\geq \quad \|X-Y+\delta(Y-p)\|^2 + \delta\varepsilon^2\,. \quad (10)
\end{aligned}$$

This in turn implies

$$\begin{aligned}
\|X-Y+\delta(X-p)\|-\|X-Y+\delta(X-Y)\| \quad &\geq \\
\geq \quad & \frac{\delta\varepsilon^2}{\|X-Y+\delta(X-p)\|+\|X-Y+\delta(X-Y)\|} \\
\geq \quad & \frac{\delta\varepsilon^2}{(2+\delta)\operatorname{diam} D+\delta\sup\{\operatorname{dist}(p,w):w\in D\}}\,. \qquad (11)
\end{aligned}$$

Thus for all small enough $\delta > 0$ (bounded by (9)) there is a constant $a = a(\varepsilon, \delta, p, D) > 0$ such that $\mathrm{d}\,\Phi(X,Y))^2 > a\,\mathrm{d}\,t$ while $\|X-Y\| > \varepsilon$, no matter what co-adapted coupling is employed. This establishes requirement 1 of Lemma 5. Note that we have not yet used the condition that ∂D be free of line segments.

The second step is to establish Lemma 5 requirement 2 **(drift bounded from above)**. From (5),

$$\begin{aligned}
&\operatorname{Drift}\mathrm{d}\,\Phi(X,Y) \quad = \\
&\quad \langle X-Y+\delta(X-p),\nu(X)\rangle\,\mathrm{d}\,L^X+\langle Y-X-\delta(Y-p),\nu(Y)\rangle\,\mathrm{d}\,L^Y+(n-\tfrac{1}{2}\operatorname{tr}(\mathbb{J}+\mathbb{J}^\top)\,\mathrm{d}\,t\,. \qquad (12)
\end{aligned}$$

Since $\operatorname{tr}(\mathbb{J}+\mathbb{J}^\top)\geq -2n$, requirement 2 of Lemma 5 is established with $b = 2n$ if we can show, for x, y in $\overline{D}$ with $\|x-y\|\geq\varepsilon$,

$$\begin{aligned}
\langle x-y+\delta(x-p),\nu(x)\rangle \leq 0 \qquad & \text{when } x\in\partial D\,,\\
\langle y-x-\delta(y-p),\nu(y)\rangle \leq 0 \qquad & \text{when } y\in\partial D\,.
\end{aligned} \qquad (13)$$

Both inequalities hold for all small enough $\delta > 0$ depending on ε, p, and the geometry of D. Consider for example the second inequality in (13). The Euclidean set

$$K \;=\; \{(x,y,\nu): x\in\overline{D}, y\in\partial D, \nu \text{ is unit inward-pointing normal to } \partial D \text{ at } y\}\,,$$

is closed and bounded hence compact. By convexity of D the inner product $\langle y-x,\nu\rangle$ is non-positive on the subset K and can vanish on K only when $x = y$ or the segment between x and y lies in ∂D. Moreover this inner product is a continuous function on K. But ∂D contains no line segments and so $\langle y-x,\nu\rangle$ is negative everywhere on the compact set $K\setminus\{(x,y,\nu):\|x-y\|<\varepsilon\}$, and therefore must satisfy a negative upper bound. Thus the second inequality of (13) also holds when $\|x-y\|\geq\varepsilon$, for all small enough δ depending on ε, p, and the geometry of D. The first inequality follows similarly.

Hence both requirements of Lemma 5 apply for $\Phi = V_{p,\delta}$ for any fixed $\varepsilon > 0$ once δ is small enough (depending on ε and the geometry of D). It follows from Lemma 4 that any co-adapted coupling X, Y eventually attains $\operatorname{dist}(X,Y)\leq\varepsilon$ for any $\varepsilon > 0$, and hence no co-adapted coupling can be shy. □

Theorem 8. *Let D be a bounded convex planar domain. Then no co-adapted coupling of reflected Brownian motions in D can be shy.*

Proof. The proof strategy is the same as for Theorem 7: exhibit, for any fixed $\varepsilon > 0$, a function Φ satisfying the requirements of Lemma 5. However the function will now be the minimum $\Phi = \Phi_1\wedge\ldots\wedge\Phi_k$ of a finite number of functions Φ_i of the form $V_{p,\delta}$ or a mild generalization thereof. The Tanaka formula (Revuz and Yor, 1991, VI Section 1.2) shows that if each Φ_i satisfies requirement 1 of Lemma 5, and satisfies requirement 2 on $\Phi_i = \Phi$, then Φ satisfies both requirements and so we may apply Lemmas 5 and then 4 as above.

The mild generalization modifies $V_{p,\delta}(x,y)$ by a small perturbation. This is required in order to deal with the possibility that ∂D contains parallel line segments.

$$\begin{aligned}\widetilde{V}_{p,\delta,S}(x,y) &= \tfrac{1}{2}\|x-y\|^2 + \tfrac{\kappa}{2}\left(\|x-p\|^2 - \|y-p\|^2\right)\\ &= \tfrac{1}{2}\|x-y\|^2 + \kappa\langle x-y, \tfrac{x+y}{2}-p\rangle\,,\\ &\text{where now } \kappa = \delta\exp\left(-\tfrac{1}{S}\|\tfrac{x+y}{2}-p\|\right)\,. \end{aligned} \tag{14}$$

Thus the asymmetry of the hyperbolic perturbation $\widetilde{V}_{p,\delta,S}$ now depends on the distance of the mid-point $\frac{x+y}{2}$ from the pole p.
The first step is to establish Lemma 5 requirement 1 **(volatility bounded from below)**. The argument runs much as for Theorem 7, but is further complicated by the need to deal with parallel line segments in ∂D. For convenience we introduce the notation

$$Z \quad = \quad \tfrac{X+Y}{2} - p\,, \qquad\qquad \mathbf{e} \quad = \quad Z/\|Z\|\,. \tag{15}$$

Thus (using the fact that $\mathbb{J}$ is a contraction)

$$\begin{aligned}(\mathrm{d}\,\widetilde{V}_{p,\delta,S}(X,Y))^2/\,\mathrm{d}\,t \quad &=\\ \|X-Y+\kappa(X-p-\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e}) &- \mathbb{J}(X-Y+\kappa(Y-p+\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e}))\|^2 +\\ &+\|\,\mathbb{K}(X-Y+\kappa(Y-p+\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e}))\|^2\\ \geq \quad \Big(\|X-Y+\kappa(X-p-\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e})\| &- \|X-Y+\kappa(Y-p+\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e})\|\Big)^2\,.\end{aligned}$$

Now consider the difference of the squared norms of the summands: if

$$S \quad > \quad \sup\{\mathrm{dist}(p,w) : w\in D\}\,, \tag{16}$$

$$\kappa \;\leq\; \delta \quad < \quad \varepsilon/\sup\{\mathrm{dist}(p,w) : w\in D\} \tag{17}$$

(so that in particular $1-\frac{\|Z\|}{S}>0$) then

$$\begin{aligned}\|X-Y+\kappa(X-p-\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e})\|^2 &- \|X-Y+\kappa(Y-p+\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e})\|^2\\ &= 2\kappa\langle X-Y+\kappa Z, X-Y-\tfrac{1}{S}\langle X-Y,Z\rangle\,\mathbf{e}\rangle\\ &= 2\kappa\left(\|X-Y\|^2 - \tfrac{\|Z\|}{S}\langle X-Y,\mathbf{e}\rangle^2 + \kappa\|Z\|(1-\tfrac{\|Z\|}{S})\langle X-Y,\mathbf{e}\rangle\right)\\ &\geq 2\kappa\|X-Y\|^2\left(1-\tfrac{\|Z\|}{S}\right) - 2\kappa\left(\kappa\|Z\|(1-\tfrac{\|Z\|}{S})|\langle X-Y,\mathbf{e}\rangle|\right)\\ &\geq 2\kappa\|X-Y\|^2\left(1-\tfrac{\|Z\|}{S}\right)\left(1-\kappa\|Z\|\tfrac{|\langle X-Y,\mathbf{e}\rangle|}{\|X-Y\|^2}\right)\\ &\geq \quad 2\kappa\|X-Y\|^2\left(1-\tfrac{\sup\{\mathrm{dist}(p,w):w\in D\}}{S}\right)\left(1-\kappa\frac{\sup\{\mathrm{dist}(p,w):w\in D\}}{\varepsilon}\right)\,.\end{aligned}$$

It follows from inequalities (16) and (17) that we obtain a positive lower bound on

$$\|X-Y+\kappa(X-p-\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e})\| - \|X-Y+\kappa(Y-p+\tfrac{1}{2S}\langle X-Y,Z\rangle\,\mathbf{e})\|$$

subject to the further condition (required to obtain a lower bound on each of the two norms in the difference above)

$$\kappa \;\leq\; \delta \quad < \quad \varepsilon/\left(\sup\{\mathrm{dist}(p,w) : w\in D\} + \tfrac{1}{2}\,\mathrm{diam}(D)\right)\,. \tag{18}$$

Now, for a finite set of poles $p_i^\pm$, to be determined below in the "localization argument", consider

$$\Phi(x,y) = \bigwedge_{i=1}^{m}\left(\widetilde{V}_{p_i^+,\delta,S}(x,y)\wedge\widetilde{V}_{p_i^+,\delta,S}(y,x)\wedge\widetilde{V}_{p_i^-,\delta,S}(x,y)\wedge\widetilde{V}_{p_i^-,\delta,S}(y,x)\right). \tag{19}$$

Subject to S and κ satisfying the conditions (16, 17, 18) as p runs through the poles $p_j^\pm$ it then follows from the Tanaka formula that Φ satisfies requirement 1 of Lemma 5.

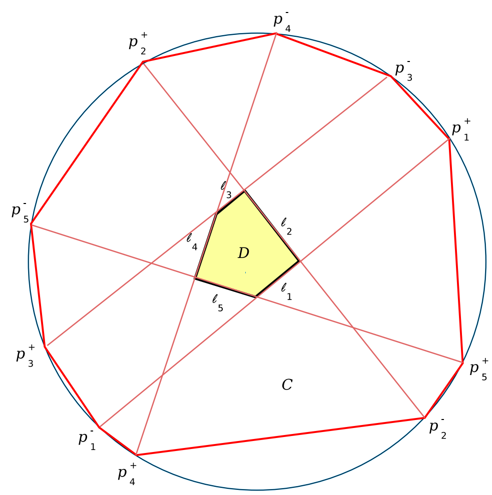

Figure 1: Circle C centred in D of large radius R, with intersection points produced by extending the maximal line segments in ∂D (for simplicity D is chosen to be a polygon). Note that ℓ_1 and ℓ_3 are parallel.

The second step is a **"localization" argument** aimed eventually at showing that requirement 2 of Lemma 5 (drift bounded from above) is satisfied on each locus $\Phi_i = \Phi$. To this end we must first specify the various poles $p_i^\pm$ involved in the different Φ_i making up $\Phi = \Phi_1 \wedge \ldots \wedge \Phi_k$ and identify the region where $\Phi_i = \Phi$ (thus, "localizing"). For fixed $\varepsilon > 0$ there can only be finitely many maximal linear segments $\ell_1, \ldots, \ell_m \subset \partial D$ of length at least ε. Fix a circle C centred in D of radius R, and locate the poles p_i^+, p_i^- at the intersection points on C of the line defined by ℓ_i, sign $\pm$ chosen according to orientation (Figure 1). Here R must be chosen to be large enough to fulfil asymptotics given below at (21): in addition we require that $R > \operatorname{diam}(D)\csc(\phi)$, where 3ϕ is the minimum modulo π of the non-zero angles between lines ℓ_i, ℓ_j. The rationale for this is that if no other line segments are parallel to a given ℓ_i, and if $x, y \in \ell_i$, then the term corresponding to ℓ_i (*via* the poles $p_i^\pm$) in the minimum (19) is then the unique minimizer. For example $\widetilde{V}_{p_i^+,\delta,S}(x,y)$ is the unique minimizer if $\kappa\langle x-y, \frac{x+y}{2} - p_i^+\rangle$ is the unique minimum of the corresponding inner products. Hence $R > \operatorname{diam}(D)\csc(\phi)$ suffices to localize in the absence of parallelism, by a simple geometric argument indicated in Figure 2. This argument uses the remark (established by calculus) that

$$\exp\left(-\frac{1}{S}\|\tfrac{x+y}{2} - p_i^+\|\right)\langle x-y, \frac{x+y}{2} - p_i^+\rangle$$

is the minimum if no other $p_j^\pm$ is separated from D by the perpendicular to ℓ_i at p_i^+ (which follows from the constraint of (16), which we have required for all poles p).

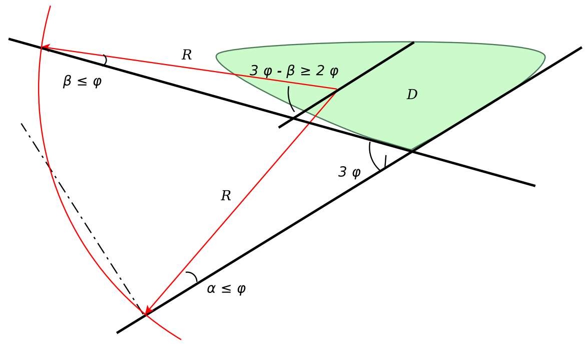

Figure 2: Let 3ϕ be the minimum modulo π of the non-zero angles between line segments ℓ_i, ℓ_j. If $R > \operatorname{diam}(D)\csc(\phi)$ and ℓ_i is parallel to no other line segments then the poles p_i^+, p_i^- for a given ℓ_i can be seen to supply the minimum in (19) when $x, y \in \ell_i$.

In case some of the ℓ_i are parallel, then we must use the particular features of $\widetilde{V}_{p,\delta,S}$ as opposed to $V_{p,\delta}$, taking into account the expression (14) for κ. Suppose that $x, y \in \ell_i$ and ℓ_i is parallel to ℓ_j. Establish cartesian coordinates (u, v) centred at $\frac{x+y}{2}$, for which ℓ_i lies on the u-axis. Without loss of generality suppose that ℓ_j lies on the locus $v = h$. Suppose the circle C is centred at (u_0, v_0) lying between ℓ_i and ℓ_j (Figure 3).

We need to show that, for all large enough R and all other poles $p_j^\pm$ for which $\langle x-y, \frac{x+y}{2} - p_j^\pm\rangle$ has the same sign as $\langle x-y, \frac{x+y}{2} - p_i^+\rangle$,

$$\frac{\exp\left(-\frac{1}{S}\|\frac{x+y}{2} - p_i^+\|\right)\langle x-y, \frac{x+y}{2} - p_i^+\rangle}{\exp\left(-\frac{1}{S}\|\frac{x+y}{2} - p_j^\pm\|\right)\langle x-y, \frac{x+y}{2} - p_j^\pm\rangle} \quad > \quad 1\,, \tag{20}$$

and we now show that this will be the case if $1 < \sigma = S/R < 2$ for large enough R (note that once R is large enough this is compatible with (16), which is our other requirement on R).

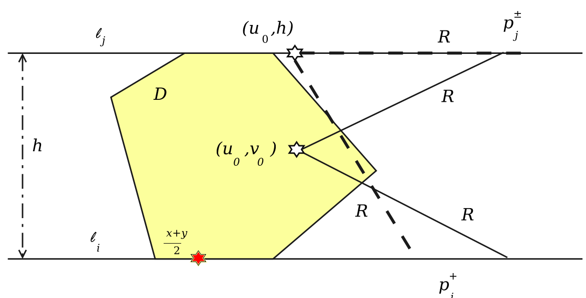

Figure 3: Illustration of the movement of poles p_i^+, $p_j^\pm$ when the centre of C is moved perpendicularly to the parallel line segments ℓ_i, ℓ_j.

First note that it suffice to consider the case when v_0 is as large as possible. For if the centre of C is moved say from (u_0, v_0) to (u_0, h) then the pole $p_j^\pm$ is moved further out on ℓ_j, while the pole

p_i^+ is brought closer in on ℓ_i (see Figure 3). Calculus shows that

$$u \exp\left(-\frac{1}{S}\sqrt{u^2+v^2}\right)$$

is increasing in u for $0 < u < S\sqrt{\frac{1+\sqrt{1+4v^2/S^2}}{2}}$ (true for all feasible moves when $\sigma = S/R > 1$, since $R > \operatorname{diam}(D)\csc(\phi)$ and $0 < \phi < \pi/3$), so it follows that such a move will increase the denominator and decrease the numerator of (20).

We now establish asymptotics which are uniform in $(u_0, v_0) \in D$; indeed, uniformly for $-\operatorname{diam}(D) \leq u_0 \leq \operatorname{diam}(D)$ and holding $\sigma = S/R$ fixed,

$$\begin{aligned}
\frac{\exp\left(-\frac{1}{S}\|\frac{x+y}{2}-p_i^+\|\right)\langle x-y, \frac{x+y}{2}-p_i^+\rangle}{\exp\left(-\frac{1}{S}\|\frac{x+y}{2}-p_j^\pm\|\right)\langle x-y, \frac{x+y}{2}-p_j^\pm\rangle} \quad &\geq \\
\geq \quad \frac{\sqrt{R^2-h^2}+u_0}{R+u_0} \times \frac{\exp\left(-\frac{1}{S}(\sqrt{R^2-h^2}+u_0)\right)}{\exp\left(-\frac{1}{S}\sqrt{(R+u_0)^2+h^2}\right)} \\
= \left(1-\frac{R-\sqrt{R^2-h^2}}{R+u_0}\right) \times \\
\times \exp\left(\frac{1}{R\sigma}\left((R-\sqrt{R^2-h^2})+(\sqrt{(R+u_0)^2+h^2}-(R+u_0))\right)\right) \\
\geq \left(1-\tfrac{R}{R+u_0}(\tfrac{h^2}{2R^2}+o(R^{-2}))\right) \times \\
\times \exp\left(\frac{1}{\sigma}\left(\tfrac{h^2}{2R^2}+o(R^{-2})+\tfrac{R+u_0}{R}(\tfrac{h^2}{2(R+u_o)^2}+o((R+u_0)^{-2}))\right)\right) \\
= \quad 1+\left(\tfrac{1}{\sigma}-\tfrac{1}{2}\right)\tfrac{h^2}{R^2}+o(\tfrac{1}{R^2}). \quad (21)
\end{aligned}$$

It follows from these uniform asymptotics that if $1 < \sigma < 2$ then (20) is satisfied for all large enough R.

As a consequence of these considerations, and bearing in mind the continuity properties of the criterion ratio on the left-hand side of (20), if we fix $S/R \in (1,2)$ and R large enough then (noting $\|x-y\| > \varepsilon$) it follows that for all sufficiently small $\eta > 0$, if $x, y \in D \cap (\ell_i \oplus \operatorname{ball}(0,\eta))$ and one of $x, y \in \partial D$ then the active component of (19) is the one involving ℓ_i *via* the poles $p_i^\pm$.

The final step is to establish Lemma 5 requirement 2 **(drift bounded from above)**. When $p^\pm$ is the active pole, the drift is given by

$$\begin{aligned}
&\left\langle X-Y+\kappa(X-p^\pm)-\tfrac{\kappa}{2S}\langle X-Y,Z\rangle\,\mathbf{e}, \nu(X)\right\rangle \mathrm{d}\,L^X + \\
&\qquad + \left\langle Y-X-\kappa(Y-p^\pm)-\tfrac{\kappa}{2S}\langle X-Y,Z\rangle\,\mathbf{e}, \nu(Y)\right\rangle \mathrm{d}\,L^Y \\
&\qquad\qquad + (n-\tfrac{1}{2}\operatorname{tr}(\mathbb{J}+\mathbb{J}^\top))\,\mathrm{d}\,t
\end{aligned}$$

(using the notation of the proof of Theorem 8). In order to establish requirement 2 of Lemma 5, it suffices to show non-positivity of

$$\left\langle X-Y+\kappa(X-p^\pm)-\tfrac{\kappa}{2S}\langle X-Y,Z\rangle\,\mathbf{e}, \nu(X)\right\rangle$$

when $X \in \partial D$, and of

$$\left\langle Y - X - \kappa(Y - p^{\pm}) - \tfrac{\kappa}{2S}\langle X - Y, Z\rangle\,\mathbf{e}, \nu(Y)\right\rangle$$

when $Y \in \partial D$. Bearing in mind that $\|X - Y\| \leq \operatorname{diam}(D)$ and $\kappa \leq \delta$, if

$$\delta \quad < \quad \frac{\xi}{\operatorname{diam}(D) + R} \times \frac{1}{1 + \frac{1}{2S}\operatorname{diam}(D)} \tag{22}$$

then this follows directly when $\langle X - Y, \nu(X)\rangle > \xi$ (in case $X \in \partial D$) or when $\langle Y - X, \nu(Y)\rangle > \xi$ (in case $Y \in \partial D$).

It remains to consider the case when this does not happen. By choosing ξ small enough, we may then ensure that the two Brownian motions are both close to the same segment portion of the boundary. For the function

$$\frac{\langle y - x, \nu\rangle}{\|x - y\|}$$

is continuous on the closed and bounded (and therefore compact) Euclidean subset

$$H \quad = \quad \left\{(x, y, \nu) \,:\, x \in D, y \in \partial D, \|x - y\| \geq \varepsilon, \nu \text{ normal at } y\right\},$$

and vanishes only on $\bigcup_i \{(x, y, \nu) \in H : x, y \in \ell_i\}$. Consequently for any $\eta > 0$ we can find $\xi > 0$ such that

$$\langle y - x, \nu(y)\rangle \quad < \quad \|x - y\|\xi$$

forces x, $y \in \ell_i \oplus \operatorname{ball}(0, \eta)$ for some i. If η is chosen as above then this in turn forces $p_i^{\pm}$ to be the pole for x, y.

We now argue for the case when $y \in \partial(D)$; the case when $x \in \partial(D)$ is similar. We can choose $\eta > 0$ as small as we wish: we therefore require that $\sqrt{1 - \eta^2/\varepsilon^2} - \eta/\varepsilon > 0$.

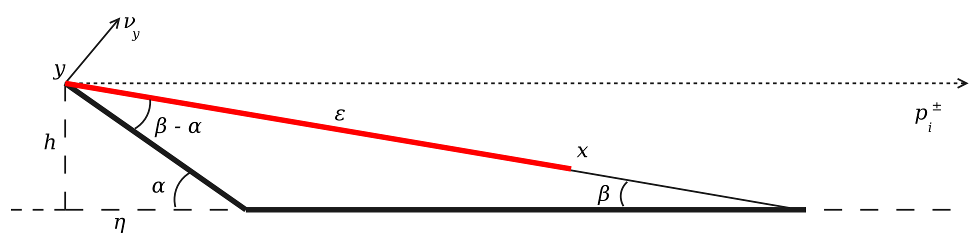

Figure 4: Geometric construction underlying analysis of singular drift when $y \in \partial D$ and $\langle y - x, \nu(y)\rangle < \|x - y\|\xi$.

Suppose then that y is at perpendicular distance $h < \eta$ from the line through ℓ_i (Figure 4). If x is further away from ℓ_i than y, then the singular drift will certainly be non-negative once (22) is satisfied, since the segment $y - x$ then makes a smaller angle with $\nu(y)$ than does either $y - p_i^{\pm}$ or $\frac{x+y}{2} - p_i^{\pm}$. Otherwise (using the angle notation indicated in Figure 4) we require non-negativity of

$$\varepsilon \sin(\alpha - \beta) - \delta \times (\operatorname{diam}(D) + R)(1 + \frac{1}{2S}\operatorname{diam}(D)) \sin\alpha\,.$$

Hence we require

$$
\begin{aligned}
\delta \quad < \quad & \frac{\varepsilon}{(\operatorname{diam}(D)+R)(1+\frac{1}{2S}\operatorname{diam}(D)}\frac{\sin(\alpha-\beta)}{\sin\alpha} \\
= \quad & \frac{\varepsilon}{(\operatorname{diam}(D)+R)(1+\frac{1}{2S}\operatorname{diam}(D))}(\cos\beta-\sin\beta\cot\alpha) \\
\leq \quad & \frac{\varepsilon}{(\operatorname{diam}(D)+R)(1+\frac{1}{2S}\operatorname{diam}(D))}\left(\sqrt{1-\frac{h^2}{\varepsilon^2}}-\frac{h}{\varepsilon}\frac{\eta}{h}\right) \\
\leq \quad & \frac{\varepsilon}{(\operatorname{diam}(D)+R)(1+\frac{1}{2S}\operatorname{diam}(D))}\left(\sqrt{1-\frac{\eta^2}{\varepsilon^2}}-\frac{\eta}{\varepsilon}\right). \qquad (23)
\end{aligned}
$$

Thus for all sufficiently small δ and sufficiently large R the singular drift is non-positive (the case of $x \in \partial D$ following by the same arguments) and so the drift is bounded above as required, thus completing the proof.

□

4 Conclusion

The above shows how rather direct potential-theoretic methods permit the deduction of non-existence of shy couplings for reflected Brownian motion in bounded convex domains, so long as either (a) the domain boundary contains no line segments (Theorem 7) or (b) the domain is planar (Theorem 8). One is immediately led to the conjecture that there are no shy couplings for reflected Brownian motion in *any* bounded convex domains. In the planar case one can extend the substantial linear portions of the boundary to produce a finite set of poles $p^{\pm}$, leading in turn to the key function Φ expressed as a minimum of a finite number of simpler functions. Similar constructions will in the general case lead to a continuum of possible poles, contained in an essential continuum of different hyperplanes, and this variety of poles is an obstruction to generalization of the crucial localization analysis in the proof of Theorem 8. Careful geometric arguments allow progress to be made in the case of bounded convex polytopes, for which attention can be confined to a finite number of hyperplanes and hence to sets of poles forming hypercircles *via* intersection of a large hypersphere with the hyperplanes. However even in this limited special case tedious arguments are required in order to overcome problems arising from intersections of the hypercircles, and we omit the details.

Further variants on the general theme of shy-ness are possible. For example, it is not hard to use the comparison techniques described in Jost et al. (1998, Chapter 2) to show the following: if $\mathbb{M}$ is a Riemannian manifold with sectional curvatures all bounded above by a finite constant κ^2, then there exist ε-shy couplings of Brownian motion on $\mathbb{M}$ for all ε satisfying

$$
\varepsilon \quad < \quad \min\left\{\frac{\pi}{2\kappa},\ \text{injectivity radius of } \mathbb{M}\right\}.
$$

These couplings are geometric versions of the perverse coupling described in Section 1.

However the major challenge remains the conjecture of Benjamini et al. (2007, Open problem 4.5(ii)), who ask whether there is any simply connected planar domain which supports a shy coupling of reflected Brownian motions. The present work brings us close to understanding shy coupling for convex domains; progress in resolving the Benjamini et al. (2007) conjecture requires development of completely new techniques not dependent on convexity at all.

Acknowledgements

I am grateful to Chris Burdzy, Jon Warren, and Saul Jacka for very helpful conversations.

References

R. Atar and K. Burdzy. On nodal lines of Neumann eigenfunctions. *Electronic Communications in Probability*, 7:129–139, 2002. MR1937899

R. Atar and K. Burdzy. On Neumann eigenfunctions in lip domains. *Journal of the American Mathematical Society*, 17(2):243–265 (electronic), 2004. MR2051611

R. F. Bass and E. P. Hsu. The semimartingale structure of reflecting Brownian motion. *Proc. Amer. Math. Soc.*, 108(4):1007–1010, 1990. MR1007487

I. Benjamini, K. Burdzy, and Z.-Q. Chen. Shy couplings. *Probability Theory and Related Fields*, 137 (3-4):345–377, March 2007. MR2278461

K. Burdzy and W. S. Kendall. Efficient Markovian couplings: examples and counterexamples. *The Annals of Applied Probability*, 10(2):362–409, May 2000. MR1768241

M. Émery. *Stochastic calculus in manifolds*. Universitext. Springer-Verlag, Berlin, 1989. With an appendix by P.-A. Meyer. MR1030543

M. Émery. On certain almost Brownian filtrations. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(3): 285–305, 2005.

K. Itô. Stochastic differentials. *Applied Mathematics and Optimization*, 1:374–381, 1975. MR0388538

J. Jost, W. S. Kendall, U. Mosco, M. Röckner, and K. T. Sturm. *New Directions in Dirichlet Forms*, volume 8 of *Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 1998. MR1652277

W. S. Kendall. Probability, convexity, and harmonic maps with small image I: Uniqueness and fine existence. *Proceedings of the London Mathematical Society (Third Series)*, 61:371–406, 1990. MR1063050

W. S. Kendall. Convexity and the hemisphere. *The Journal of the London Mathematical Society (Second Series)*, 43:567–576, 1991. MR1113394

W. S. Kendall. Coupling all the Lévy stochastic areas of multidimensional Brownian motion. *The Annals of Probability*, 35(3):935–953, May 2007. MR2319712

M. N. Pascu and M. E. Gageonea. On a conjecture of Laugesen and Morpurgo, 2008. URL `http://arxiv.org/abs/0807.4726`.

D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991.

H. Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.*, 9(1):163–177, 1979. MR0529332